NBER WORKING PAPER SERIES

# QUANTUM ECONOMIC ADVANTAGE

Francesco Bova
Avi Goldfarb
Roger G. Melko

Working Paper 29724
http://www.nber.org/papers/w29724

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
February 2022

Roger G. Melko is supported by the Natural Sciences and Engineering Research Council of Canada (NSERC), Canada Research Chair (CRC) program, and the Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Colleges and Universities. Avi Goldfarb is supported by the Social Sciences and Humanities Council of Canada. His full disclosure statement is available at www.avigoldfarb.com/disclosure. Francesco Bova is supported by a CPA Ontario Center for Accounting Innovation Research Grant. We thank seminar participants at the MIT Digital Initiative for helpful comments. We are grateful to participants in the Creative Destruction Lab quantum stream who helped us understand the business aspects of quantum computing. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

Quantum Economic Advantage
Francesco Bova, Avi Goldfarb, and Roger G. Melko
NBER Working Paper No. 29724
February 2022
JEL No. L63,M15,O3

**ABSTRACT**

A quantum computer exhibits a quantum advantage when it can perform a calculation that a classical computer is unable to complete. It follows that a company with a quantum computer would be a monopolist in the market for solving such a calculation if its only competitor was a company with a classical computer. Conversely, economic outcomes are unclear in settings where quantum computers do not exhibit a quantum advantage. We model a duopoly where a quantum computing company competes against a classical computing company. The model features an asymmetric variable cost structure between the two companies and the potential for an asymmetric fixed cost structure, where each firm can invest in scaling its hardware to expand its respective market. We find that even if: 1) the companies can complete identical calculations, and thus there is no quantum advantage, and 2) it is more expensive to scale the quantum computer, the quantum computing company can not only be more profitable but also invest more in market creation. The results suggest that quantum computers may not need to display a quantum advantage to be able to generate a quantum economic advantage for the companies that develop them.

Francesco Bova
Rotman School of Management
University of Toronto
105 St George St
Toronto, ON M5S3E6
Canada
Francesco.Bova@Rotman.Utoronto.Ca

Roger G. Melko
Department of Physics and Astronomy
University of Waterloo
200 University Avenue West
Waterloo, ON N2L3G1
Canada
rmelko@perimeterinstitute.ca

Avi Goldfarb
Rotman School of Management
University of Toronto
105 St. George Street
Toronto, ON M5S 3E6
CANADA
and NBER
agoldfarb@rotman.utoronto.ca

# 1 Introduction

We live in an exciting era of optimism in the field of quantum computing where the first claims of quantum *supremacy* or *advantage*[1] are being made on a number of innovative quantum devices. Very simply, we say that a quantum computer has a *quantum advantage* if it can perform some calculation, no matter how arbitrary, that a conventional (or "classical") computer cannot complete. Demonstrating quantum advantage has been a major pursuit of a large community of scientists and engineers working on different types of quantum computers in academia and industry, and will be considered a watershed moment in the history of science when and if it is achieved (National Academies of Sciences and Medicine 2019). This pursuit has also led to a growing discussion of the business opportunities that may arise from quantum computers (Ruane, McAfee, and Oliver 2022, Bova, Goldfarb, and Melko 2021) and quantum's potential to disrupt or replace classical computing (Cusumano 2018, Yang, Chesbrough, and Hurmelinna-Laukkanen 2021).

In 2019, Martinis' landmark experiment at Google was the first to make a strong claim for quantum advantage (Arute et al. 2019). Using a programmable 53-qubit superconducting qubit device called Sycamore, the experiment produced a probability distribution by sampling a random quantum circuit that was thought to be impossible to simulate classically at the time. Google's claim motivated several groups to repeat the equivalent classical calculation using new algorithms and powerful hardware, thereby moving the bar which defined quantum advantage for that particular calculation (Pan and Zhang 2021; Huang et al. 2021). Subsequently, another collaboration using a 56-qubit superconducting device called Zuchongzhi seemingly put the calculation out of reach for classical computers once again (Wu et al. 2021).

The example above illustrates the challenges in illustrating a quantum advantage. These challenges arise in part because the benchmark for quantum advantage keeps moving. It is also important to note that the Sycamore and Zuchongzhi calculations discussed above, while

[1]We adopt the latter terminology in this article (see Mueck, Palacios-Berraquero, and Persaud 2020).

certainly groundbreaking achievements, involve random quantum circuits that have no immediate practical applications. It stands to reason that finding a business application which can display quantum advantage may be even more challenging to do. With uncertainty about the ability of quantum computers to generate a quantum advantage over classical approaches on problems of practical interest, natural questions arise as to whether a quantum computer can still generate economic value without being able to generate a quantum advantage.

In this paper, we argue that quantum computers can still generate economic value even when they do not provide a quantum advantage over classical computers. This outcome arises because there are asymmetries in the cost structure between the classical and quantum computer. To illustrate the intuition, we consider a strategic game, using a duopoly model where a quantum computing company and classical computing company compete. In this way, our paper builds on a well-established literature in information systems that uses game theoretic models to understand how technological change might impact new and existing industries (e.g. Bakos and Nault 1997 on electronic networks, Zhu and Iansiti 2012 and Adner, Chen, and Zhu 2020 on platforms, Niculescu and Wu 2014 on zero marginal cost software, and Dellarocas, Katona, and Rand 2013 on hyperlinks).

In the model, each firm's cost structure is influenced by two different factors, based on differences between quantum and classical computers. The first factor is the ability of quantum algorithms to speed up certain processes relative to algorithms that are run on classical hardware, where the expected efficiency brought on by quantum algorithms leads to a variable cost advantage for quantum computers. We use the functional form provided by Grover's algorithm in our analysis, specifically a quadratic speedup (Grover 1996).

The second factor is the cost to scale each hardware, where the more qubits (bits) a quantum (classical) computer has, the larger the problems it can solve and thus the larger the market it can address. Notably, it is currently less expensive to scale a classical computer than it is to scale a quantum computer (e.g. Moore 2006, Ball 2021). It follows that the ability for classical computers to scale more cheaply may lead to a cost advantage for classical computers.

In our model, customers are purchasing a solution to an intractable problem. The offerings provided by the quantum computing company and classical computing company can be differentiated or homogeneous. When the products are perfectly differentiated, one computer can solve a problem that the other computer cannot solve, and vice versa. For example, a sufficiently coherent quantum computer running a quantum algorithm could solve certain problems that classical computers cannot solve in human lifetimes (e.g., the factoring of very large numbers into their respective primes (Shor 1999)). In this setting, the quantum computing company is the monopolist in the market to solve the problem that generated the quantum advantage. When the firms' offerings become more homogeneous, both types of computers can output the same solution to solve the same problem. When the market deems each company's offerings to be perfect substitutes for each other, the quantum and classical computer offer identical offerings. This represents the case in which the quantum computer does not exhibit a quantum advantage over its classical competitor, in the sense that it is feasible to use either computer to solve the problem.

Regardless of the level of differentiation between the firms' offerings, there may still be asymmetric costs to generating a solution across the two firms. As we note above, the principal factor that impacts the variable cost structure of each company's offering is the speed in which the solution is generated. In our simplified model, we model variable costs as the amount of time it takes to output a solution.

Additionally, there is a fixed cost component to each firm's profit function. Each firm can choose to invest a fixed amount in to market creating investments. These market creating investments lead to improvements in each firm's underlying hardware which allow their respective computers to solve larger, more intractable problems. For the quantum computing company, these investments may represent the development of higher quality qubits, improvements in entanglement, and enhancements in error correction. Investments in these areas should allow the quantum computer to solve increasingly larger and more intractable problems. Similarly, the classical computer company might make investments to build compute capabilities which will allow the classical firm to solve increasingly larger problems.[2] Taken together, as investments in each company's hardware increase, the size of the problems each company can assess become bigger, and in turn the size of each firm's addressable market also become bigger.

We model competition as a Cournot duopoly, where each firm sets quantity simultaneously. These quantities, in part, determine the price of each firm's offering. In our setting, quantities can be thought of as a commitment to the amount of computing time each company makes available to consumers to solve intractable problems. This assumption implies that each firm makes a finite amount of computing time available to prospective consumers. This implication is consistent with observations from the current computing landscape, as we currently do not have ubiquitous accessibility to either quantum computers or classical supercomputers to solve these large intractable problems.[3]

As the cost to solving challenging problems is decreasing in both the speed up associated with certain algorithms (which may favor quantum computers) and the size of and ability to scale the hardware (which may currently favor classical computers), it is not clear which of the two architectures will ultimately have a lower cost structure in aggregate. In this way, we focus on two forces separate from quantum advantage as typically defined. First, our model highlights that, for a given scale, quantum computers perform some calculations faster than classical computers, even though such calculations are feasible on both types of computer. Second, our model highlights that current quantum computers are more expensive to scale than classical computers. These two forces determine the market opportunity for quantum and classical computing.

[2]For example, the increases in the amount of compute used in A.I. training (https://openai.com/blog/ai-and-compute/) and NVIDIA's investments in GPU scaling (https://developer.nvidia.com/blog/scaling-out-the-deep-learning-cloud-efficiently/).

[3]Our modeling assumptions on fixed capacity, duopoly competition, and differences in variable costs reflect similar assumptions in other information systems contexts, such as Choudhary and Vithayathil 2013 and Fazli, Sayedi, and Shulman 2018 on cloud computing and Abhishek, Jerath, and Zhang 2016 and Zhang 2009 on online retail.

Our analysis illustrates that even when there is no quantum advantage and it is cheaper to scale the classical computer, the quantum computing company can still have a greater incentive to invest in building its market and can ultimately still be more profitable than the classical computing company. These outcomes arise due to the efficiency of the quantum computing company's variable cost structure. We also illustrate that the quantum computing company can still be a monopolist even if it does not exhibit a quantum advantage in settings where a classical computing company can complete a task, but not in a manner that's cost effective enough to be commercially viable. In aggregate, our results suggest that a quantum computing company can generate a quantum economic advantage without exhibiting a quantum advantage.

Finally, we also explore how advances in areas like error correction (which should make quantum computers easier to scale in the future) and the creation of quantum-inspired algorithms (which should mitigate the speed up advantage of quantum computers over classical computers for certain problems) affect our insights.

The paper proceeds as follows. In the second section we employ a model of differentiated Cournot competition and assess optimal investment and profitability outcomes. In the third section, we provide additional analysis and context for the main results. In the final section, we conclude.

## 2 A Cournot duopoly model

We model competition between a quantum computing company and a classical computing company. The competition occurs in two stages. In the first stage, each firm invests in the scale of the computer they build. In the second stage, the companies compete in quantity of computations.

As we note above, we assume that the duopoly is comprised of a company that has created a quantum computer and a company that has created a classical computer. The quantum computing company is labeled as firm 1 and the classical computing company is labeled as firm 2.

We begin by reprising the commonly cited inverse demand functions for a Cournot duopoly that are generated in Singh and Vives (1984, pg. 547) when a representative consumer maximizes its quadratic utility function.

$$p_1 = \alpha_1 - \beta_1 q_1 - \gamma q_2$$

$$p_2 = \alpha_2 - \beta_2 q_2 - \gamma q_1$$

$p_1$ and $p_2$ are the prices that each firm can charge for their respective offerings. $\alpha_1$ and $\alpha_2$ represent the intercept for each firm's respective demand. We set $\alpha_1 = a + x_1$ and $\alpha_2 = a + x_2$, where $a$ is an exogenous demand intercept which is common to both firms, and $x_1$ and $x_2$ are endogenous demand parameters that are specific to each firm, respectively. We discuss these endogenous parameters more below. We assume that $\beta_1 = \beta_2 = 1$ and thus the firms have a common slope for demand for each of their respective offerings. As in Singh and Vives, $\gamma$ is the measure of product differentiation and we assume $\gamma \in [0, 1]$.[4]

As $\gamma$ gets closer to zero, the product offerings become perfectly differentiated. A value of $\gamma = 0$ implies that the quantum computer has a quantum advantage – in other words it can complete a calculation that is effectively impossible for the classical computer. As $\gamma$ gets closer to one, the product offerings become more homogeneous. A value of $\gamma = 1$ implies that the quantum computer and classical computer are supplying identical offerings and thus, the quantum computer does not offer a quantum advantage over its classical counterpart.

In the Cournot setup, each firm sets its firm-specific quantity, $q_i$, to maximize profits (Cournot and Fisher 1929). As we note above, quantities can be thought of as the amount of computing time each company makes available to consumers to solve intractable problems. The price for each firm's offerings arise as a function of the chosen quantities. Following

[4] This demand structure is commonly used in the management and economics literature (.e.g Anand and Goyal 2009, Abhishek, Jerath, and Zhang 2016, and Bustamante and Frésard 2021.)

our assumptions above, the final inverse demand function for the quantum computing firm is $p_1 = a + x_1 - q_1 - \gamma q_2$ and the inverse demand function for the classical computing firm is $p_2 = a + x_2 - q_2 - \gamma q_1$. In other words, following standard economics, each firm's price is decreasing in the quantity it produces and the quantity produced by its competitor provided the offerings are not perfectly differentiated. The revenue generated by the quantum computing firm and the classical computing firm are $p_1 q_1$ and $p_2 q_2$, respectively.

Singh and Vives (1984) assume a constant variable cost structure for each firm. The firms in our model also have a constant variable cost structure, but the variable cost structure is asymmetric across the two firms. Each firm's variable cost structure assumptions are informed by the speed at which each firm's computer can solve a problem. As discussed above, we assume that the quantum computer can complete a process with less resources (e.g. in a timelier manner) than a classical computer by running a quantum algorithm. For the purpose of our exercise, we focus on one particular quantum algorithm, Grover's Algorithm, which allows for a quadratic speed up over classical algorithms for several types of problems related to unstructured searches (Grover 1996).

A quadratic speed up implies that the quantum computer can complete specific processes in square-root the number of steps compared to the equivalent classical algorithm. For the purpose of model tractability, we assume that the time it takes to complete a calculation is the sole driver of each firm's variable cost structure. To incorporate the impact of a quadratic speed up, we first set the classical computer's variable cost to $c^2$. The quantum computer, using Grover's algorithm, can complete a process in square root the number of steps as the classical computer which leads to the variable cost function $\sqrt{c^2} = c$. We assume that $a > c^2 > c > 1$ to ensure the quantum firm has a natural cost advantage over the classical computer due to the quadratic speed up brought on by the use of Grover's Algorithm, and that the common demand intercept is larger than either firm's variable cost. We also assume that each firm's demand intercept is sufficiently large relative to its variable cost to ensure that the resulting optimal quantities are strictly positive. Taken together, the variable cost base for the quantum company is calculated as $cq_1$ and for the classical company is $c^2 q_2$.

Next, we model each company's ability to scale its respective computer to solve more complex problems. First, we include an endogenous, convex, fixed cost investment which allows each firm to increase the size of the market its hardware has access to. Each firm's respective investment positively impacts their respective demand intercepts. The fixed cost investment is $B_1 x_1^2/2$ and $B_2 x_2^2/2$ for the quantum and classical firm respectively, where $x_1$ and $x_2$ are investment choice variables, and $B_1$ and $B_2$ are positive coefficients that can take different values. $B_1$ and $B_2$ taking different values reflect that it may be more costly to scale one type of computer than the other. As we note above, these investments generate an increase the size of the market (e.g., the intercept for demand) by $x_1$ and $x_2$ for the quantum and classical computing company, respectively. An increase in market size is driven by an increase in the size of the problem that each firm's hardware can assess.

If, for example, $B_1 > B_2$, then it is more costly to scale the hardware (and in turn, expand the market) for a quantum computer than it is for a classical computer. This outcome would, in part, offset the natural variable cost advantage that a quantum computer gleans via the quadratic speed up brought on by Grover's Algorithm. Taken together, while a quantum computer may be able to provide a result in a more timely manner, it may be more costly for the firm to build a computer to achieve this more timely result which may impede its ability to create or expand the market for its service. Thus the net benefits to quantum computing from a cost perspective are unclear. The full profit function for each company is represented as,

$$\pi_1 = q_1(a + x_1 - q_1 - \gamma q_2) - cq_1 - B_1 \frac{x_1^2}{2}, \tag{1}$$

$$\pi_2 = q_2(a + x_2 - q_2 - \gamma q_1) - c^2 q_2 - B_2 \frac{x_2^2}{2}. \tag{2}$$

In each profit function, the first term captures the firm's revenue, the second term captures the firm's variable cost base, and the third term captures the firm's fixed cost base. The sequence for firm decisions proceeds in two stages. In the first stage, Firm 1 sets investment $x_1$ to maximize its profits and Firm 2 sets $x_2$ to maximize its profit. In the second stage, Firm 1 sets $q_1$ to maximize its profits and Firm 2 sets $q_2$ to maximize its profits. We use backward induction to solve the program. Taking the first order condition for firm 1 (firm 2) with respect to $q_1$ ($q_2$) and then solving simultaneously yields the following optimal quantities:

$$q_1^* = \frac{-2(a-c+x_1)+(a-c^2+x_2)\gamma}{-4+\gamma^2}, \tag{3}$$

$$q_2^* = \frac{-2(a-c^2+x_2)+(a-c+x_1)\gamma}{-4+\gamma^2}. \tag{4}$$

As expected, each firm's optimal quantities are increasing in the common demand intercept and their own investments, and decreasing in competitive intensity, $\gamma$, and the competing firm's investments.

Next, we plug the optimal quantities from (3) and (4) into the profit functions in Equations (1) and (2) to generate $\pi_1(q_1^*, q_2^*)$ and $\pi_2(q_1^*, q_2^*)$. We simultaneously set $x_1$ to maximize $\pi_1(q_1^*, q_2^*)$ and $x_2$ to maximize and $\pi_2(q_1^*, q_2^*)$. Below, we define the conditions that ensure that both $\pi_1(q_1^*, q_2^*)$ and $\pi_2(q_1^*, q_2^*)$ are concave in $x_1$ and $x_2$, respectively, and that the resulting optimal investment levels $x_1^*$ and $x_2^*$ are both greater than zero.

LEMMA 1: For $\pi_1(q_1^*, q_2^*)$ and $\pi_2(q_1^*, q_2^*)$ to be concave in $x_1$ and $x_2$ respectively, and for $x_1^*$ and $x_2^* > 0$, $B_1$, $B_2$, and the intercept $a$ need to be sufficiently high.

Taking the first order condition for $\pi_1(q_1^*, q_2^*)$ ($\pi_2(q_1^*, q_2^*)$) with respect to $x_1$ ($x_2$) and then solving simultaneously yields the optimal investments $x_1^*$ and $x_2^*$,

$$x_1^* = \frac{4(a(4-B_2(-2+\gamma)^2(2+\gamma))+c(-4+B_2(-2+c\gamma)(-4+\gamma^2)))}{-8(2+B_2(-4+\gamma^2))+B_1(-4+\gamma^2)(-8+B_2(-4+\gamma^2)^2)}, \tag{5}$$

$$x_2^* = \frac{4(-4c^2-B_1c(2c-\gamma)(-4+\gamma^2)+a(4-B_1(-2+\gamma)^2(2+\gamma)))}{-8(2+B_2(-4+\gamma^2))+B_1(-4+\gamma^2)(-8+B_2(-4+\gamma^2)^2)}. \tag{6}$$

Plugging optimal quantities from Equations (3) and (4) and optimal investments from Equations (5) and (6) into the profit functions in (1) and (2) yields the following optimal profit

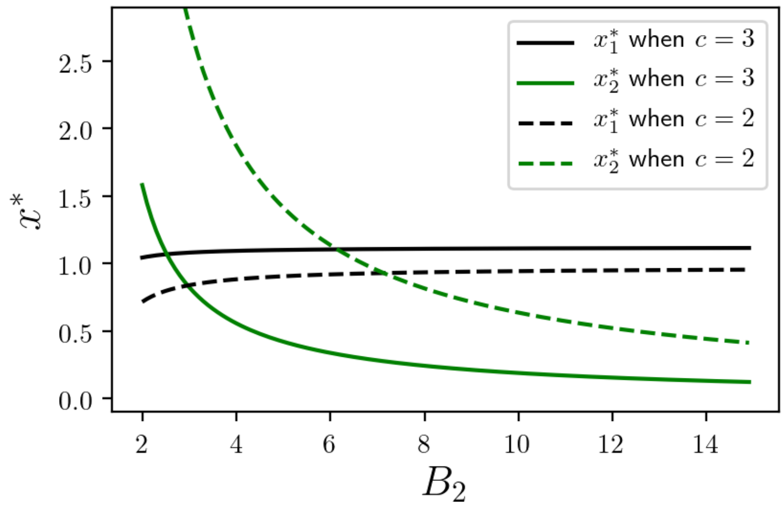


Figure 1: Optimal investments.

functions:

$$\pi_1^* = \frac{B_1(-8+B_1(-4+\gamma^2)^2)(a(-4+B_2(-2+\gamma)^2(2+\gamma))+c(4-B_2(-2+c\gamma)(-4+\gamma^2)))^2}{(-8(2+B_2(-4+\gamma^2))+B_1(-4+\gamma^2)(-8+B_2(-4+\gamma^2)^2))^2}$$

$$\pi_2^* = \frac{B_2(-8+B_2(-4+\gamma^2)^2)(4c^2+B_1c(2c-\gamma)(-4+\gamma^2)+a(-4+B_1(-2+\gamma)^2(2+\gamma)))^2}{(-8(2+B_2(-4+\gamma^2))+B_1(-4+\gamma^2)(-8+B_2(-4+\gamma^2)^2))^2}$$

*PROPOSITION 1*: Given the assumptions in Lemma 1, the quantum computing company is profitable for all $\gamma$.

Note that each firm's profits are strictly positive provided each firm's concavity conditions (defined in Lemma 1) are met. Each firm's profits are a function of $a$ (size of the common market intercept), $c$ (cost to run a program on a quantum computer), $B_1$ (investment efficiency for quantum computer), $B_2$ (investment efficiency of classical computer), and $\gamma$ (differentiation of product offerings).

When the quantum company has a quantum advantage over the classical firm, $\gamma = 0$ and the quantum company extracts monopoly rents in its market as the classical computer can't compete with the quantum computer. Perhaps more interestingly, even in cases where there is no quantum advantage (a setting where the offering are identical – i.e., $\gamma = 1$), the quantum company is still profitable even though it is not a monopolist.

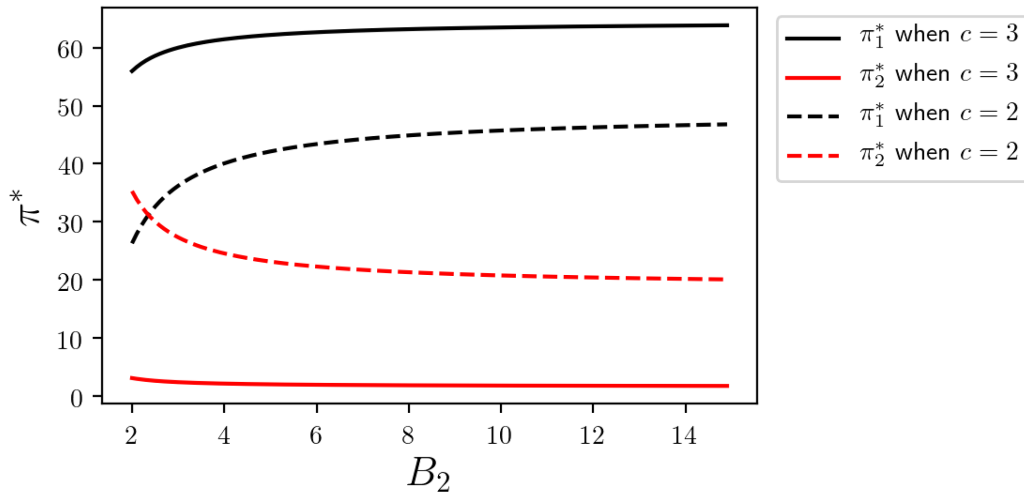


Figure 2: Optimal profits.

To provide more insight, we assess optimal investments and the resulting optimal profits numerically in Figure 1 and 2 respectively. For both Figure 1 and Figure 2, we set $a = 20$, $B_1 = 10$, and $\gamma = 1$. Notably, in a setting where $\gamma = 1$, the offerings are sufficiently homogeneous so that the quantum computer does not observe a quantum advantage. For both figures, we vary $B_2$ along the horizontal axis. Finally, we graph outcomes for both firms by varying the variable cost, $c$, to be either 2 or 3.

The figures provide some interesting insights. First, there are settings where the classical computing company is more profitable than the quantum computing company (e.g., in the parameter space where the red dashed line approaches the y-axis in Figure 2). In general, the classical firm is more profitable than the quantum firm when: (1) Variable costs are comparatively low (in some cases when $c = 2$, but never when $c = 3$), and thus the benefit of the quantum quadratic speed up is diminished (i.e., these may not be tasks that an analyst needs a quantum computer to solve); (2) The cost to scale is much lower for the classical firm than the quantum firm (i.e., $B2$ much lower than $B1$). These outcomes map well into the current computing ecosystem and provide predictions for the future. The cost to scale a classic computer is currently much lower than the cost to scale a quantum computer and thus $B_2$ is currently much lower than $B_1$. Separately, classical computers have no problems handling smaller tractable problems (i.e., problems where $c^2$ is still comparatively small). In such a setting, we would expect classical computers to be more profitable than quantum computers and this is in fact what we observe today. At some point in the future however, if the difference between $B_2$ and $B_1$ gets smaller, and the computing ecosystem attempts to tackle larger, more intractable problems (i.e., $c$ increases), then the model predicts that the quantum computing company would become the more profitable of the two companies even in settings where it does not display a quantum advantage.

Additionally, note that in the Figure 1, $x_2$ is not always greater than $x_1$ when $B_2 < B_1$. In these settings, despite it being cheaper to scale the classical computer *ex ante*, there are still circumstances where we observe greater investments in market creation by the quantum computing company (i.e.,$x_2 < x_1$ when $B_2 < B_1$), *ex post*.

*PROPOSITION 2*: There are settings where the quantum computer may invest more in market creation even in circumstances where it is less costly to scale a classical computer than a quantum computer.

This outcome arises because of the asymmetries in variable cost structures across the two companies. To illustrate the intuition behind this observation, we use a simplified model to isolate the benefit to increasing the intercept of market demand on each firm's profits, absent the impact on fixed costs. We do this by first removing market-creating investments from the optimization programs in (1) and (2) by setting $x_1 = x_2 = 0$. This yields profit functions (7) and (8) below.

$$\pi_1 = q_1(a - q_1 - \gamma q_2) - cq_1, \tag{7}$$

$$\pi_2 = q_2(a - q_2 - \gamma q_1) - c^2 q_2. \tag{8}$$

Next we optimize (7) and (8) with respect to $q_1$ and $q_2$, respectively and solve simultaneously.

Optimal quantities are as follows:

$$q_1^* = \frac{a(-2+\gamma)+c(2-c\gamma)}{(-4+\gamma^2)} \tag{9}$$
$$q_2^* = \frac{c(2c-\gamma)+a(-2+\gamma)}{(-4+\gamma^2)} \tag{10}$$

Note that $q_2^* > 0$ if $a > c(-2c+\gamma)/(-2+\gamma)$. Thus, this is a necessary condition to ensure that $q_1^*, q_2^* > 0$ for this model set up. Plugging the optimal quantities from (9) and (10), in to (7) and (8) we get the optimal profits below:

$$\pi_1^* = \frac{(a(-2+\gamma)+c(2-c\gamma))^2}{(-4+\gamma^2)^2} \tag{11}$$
$$\pi_2^* = \frac{(c(2c-\gamma)+a(-2+\gamma))^2}{(-4+\gamma^2)^2} \tag{12}$$

If we differentiate $\pi_1^*$ and $\pi_2^*$ in Equations (11) and (12) with respect to $a$, we get:

$$\frac{\partial \pi_1^*}{\partial a} = \frac{2(-2+\gamma)(a(-2+\gamma)+c(2-c\gamma))}{(-4+\gamma^2)^2}$$

and

$$\frac{\partial \pi_2^*}{\partial a} = \frac{2(c(2c-\gamma)+a(-2+\gamma))(-2+\gamma)}{(-4+\gamma^2)^2},$$

respectively. Additionally, note that $\partial \pi_1^*/\partial a > \partial \pi_2^*/\partial a > 0$ provided $q_1^*, q_2^* > 0$. Thus: 1.) the quantum computing company's profits in (11) and classical computing company's profits in (12) increase as the intercept $a$ increases (as expected), and 2.) all else equal, an increase in $a$ has a bigger impact on improving profits for the quantum computing company than on improving profits for the classical company. Said another way, all else equal, an increase in the intercept will have a smaller impact on improving the profitability for the firm with the higher variable cost structure (in this case, the classical computing firm).

Taken together, even in some settings where the cost of market creation is lower for the classical company (i.e., $B_2 < B_1$), the benefits to market creation (i.e., increase in the demand intercept) is also lower because of the classical computer's higher variable cost structure ($c^2 > c$). These competing tensions lead to instances where $x_2 < x_1$ even in cases where $B_2 < B_1$. We can conclude that having a lower cost to scaling may not necessarily lead to greater market creating investments because the classical firm is still a comparatively high variable cost producer.

Finally, as Figure 2 illustrates, there are many circumstances where the quantum computing company is more profitable than the classical computing company even when $\gamma = 1$ (i.e., where competition is at its most intense) and $B_2 < B_1$ (i.e., it is cheaper to scale the classical computer). This outcome arises because the quantum computing company's lower variable cost structure has both a direct effect on improving profitability by reducing the firm's cost base, and an indirect effect on improving profitability by generating greater benefits to investing in market creation.

# 3 Discussion

## 3.1 Quantum speed ups and quantum vs. economic advantage

In the prior section, we note that the quantum computing company is a monopolist when it has a quantum advantage over the classical computing company. Next, we illustrate a situation where the quantum computing company can still be a monopolist even when the classical computing company can also complete a task that the quantum computing company can complete.

To illustrate the point, we use a simplified example. Starting with equations (1) and (2) we first assume that the each firm in the market makes no market-creating investments, or $x_1 = x_2 = 0$. As a result, both companies face a common demand intercept of $a$. Next, we assume that $\gamma = 1$ and, as such, that both firms provide identical offerings.

With these adjusted assumptions, we provide a numerical example to illustrate how a quantum computing company can still end up as a monopolist even if it does not display a quantum advantage. First, we assume the intercept for demand is $a = 20$. Next, we note that when $c = 2$, the variable cost base for the quantum firm is $c = 2$ and for the classical firm is $c^2 = 4$. When $c = 3$, the variable cost base for the quantum firm is $c = 3$ and for the classical firm is $c^2 = 9$. Finally, if we increase the value for $c$ to $c = 5$, the variable cost base for the quantum firm is $c = 5$ and for the classical firm is $c^2 = 25$. In this last case, the variable cost for the classical firm is greater than the common intercept of the demand curve (i.e., $a = 20$). When the classical computing firm's variable cost is 25, there is no retail price that could be set for the classical computing company's service that would allow the firm to be profitable (i.e., any price less than 25 would lead to a loss and any price greater than 20 would result in no consumer demand). So we have a situation where the classical computer can complete a task in a somewhat timely manner compared to the quantum computer (i.e., 25 steps vs. 5 steps) and hence there is no quantum advantage, but where the classical computer company's offering may not be commercially viable because its variable cost base is greater than the demand intercept. This is another example of economic advantage without quantum advantage. In this example, the asymmetric cost structure would result in the quantum computing company becoming a monopolist in the market despite offering a perfect substitute to its classical competitor.

COROLLARY 1: When the classical computer can perform a calculation, but not in a timely enough manner to be commercially viable, the quantum computing company becomes a monopolist despite not exhibiting a quantum advantage

Examples of the above corollary might be most apparent in a setting where one requires computational results relatively quickly. For example, real-time transactions in financial markets or complex queries to large cloud databases may require response times in seconds, as opposed to minutes (or even hours). Quadratic speedups such as Grover's algorithm may be sufficient to deliver such speedups, as could other algorithms discussed below. Thus, in cases like this, while there may be no strict quantum advantage (in the sense that the equivalent calculation is still possible on classical hardware), quantum economic advantage is nonetheless achieved.

## 3.2 Other Quantum Algorithms and Quantum-inspired algorithms

We next explore how technological advances will affect the relative variable costs, $c$ and $c^2$. In the main model, we use Grover's algorithm to illustrate the impact of a quantum speedup on the quantum computing company's variable cost structure. With respect to Grover's algorithm, there are two important points to discuss. First, Grover's algorithm provides a *provable* improvement for a class of problems related to unstructured search when it is run on a coherent quantum computer. Second, Grover's algorithm scales polynomially with the number of qubits (i.e. $\sqrt{N}$ time for quantum versus $N$ for classical). Despite the fact that a quadratic speed up could be argued to be only a modest improvement, we have illustrated above that it is sufficient to promote a robust quantum economic advantage in certain cases. Nevertheless, our application of Grover's algorithm in the model should bias the analysis away from finding a quantum economic advantage, as there are a large variety of quantum algorithms that offer a *much larger* speed up over their classical counterparts. The most well-known of these algorithms is Shor's algorithm for factoring, which reduces the computational cost from scaling exponentially in $N$ to scaling polynomially in $N$ (Shor 1999). Shor's algorithm run on a coherent quantum computer would be so effective that it puts our standard RSA public key encryption protocols at risk! For example, factoring a 2048-bit RSA key (the size recommended by NIST) is estimated to take billions of years or more using conventional computers, whereas a fully fault-tolerant quantum computer with a sufficient number of logical qubits could complete the task in seconds to days (Van Meter and Horsman 2013).

In contrast to Grover's and Shor's algorithms, many quantum speedups have not been mathematically proven. These other quantum algorithms remain open to competition from "quantum-inspired" approaches, which are algorithmic improvements inspired by the study of quantum algorithms that can be applied to classical computers. As these new quantum-inspired algorithms continue to be developed, a quantum computer's variable cost advantage related to the timeliness of its speed up will presumably be mitigated for certain applications. For example, it was believed for some time that a particular quantum algorithm would give an exponential speedup for a certain type of machine learning problem relevant for recommendation systems (Kerenidis and Prakash 2017), like those used by Amazon and Netflix. However, Ewin Tang developed a classical algorithm that, inspired by a deep understanding of the quantum speedup, was proven to be capable of performing the same calculation on a normal computer without the need for quantum hardware (Tang 2019).

While quantum-inspired algorithms might mitigate the speed up advantage of quantum computers relative to classical computers for some problems, there are classes of problems (like those addressable by Shor's algorithm) that should only be solvable by quantum computers in a timely manner. Therefore, our model uses the mathematically proven speed-up from Grover's algorithm as the basis for the difference in variable costs and our results apply even if quantum-inspired algorithms mitigate the usefulness of other quantum algorithms.

## 3.3 Advances in Quantum Computing

It is important to note that while classical computers are currently easier to scale than their quantum counterparts, there are reasons to expect that the cost to scale each architecture will change over time. It is possible that if these costs change over time, so to will the gap between $B_1$ and $B_2$.

Historically, classical computers have scaled more efficiently over time following a pattern predicted by Moore's law (Moore 2006). Moore's law is the observation that the number of bits $n_i$ in a classical computer (transistors) doubles every 2 years:

$$n_i = n_0 2^{(y_i - y_0)/T_2} \tag{13}$$

where $n_0$ is the number of transistors in some reference year $y_0$, and $T_2 = 2$ is the doubling time. This can be inverted to say how the cost per transistor decreases as a function of year. The quantum version of Moore's law is sometimes called Rose's law, after D-Wave's founder Geordie Rose in 2002. It can be presumed to have the same form as Eq. (13), with different values of $n_0$ and $T_2$. Further simplifications in the comparison could be to assume that $T_2 = 2$ for quantum, or to say that $T_2$ for classical is slowing down as we approach the "end" of Moore's law for CMOS architectures (Track, Forbes, and Strawn 2017).

Notably, our model incorporates the cost to scale each architecture at a specific point in time. As Moore's law implies that the increase in the ability to scale classical computers will eventually slow down, it may also be reasonable to predict that the differences in scaling efficiency between a classical and quantum computer will continue to decrease over time, and that the gap between $B_1$ and $B_2$ might get smaller over time. If this future is realized, it may have a material impact on the profitability of quantum computers relative to classical computers. When the difference between $B_1$ and $B_2$ gets smaller, the difference in profitably between the quantum and classical computer becomes larger, even in settings where no quantum advantage is observed.

# 4 Conclusion

Our model has emphasized the marginal advantages of quantum computing over classical computing. The results show that it is possible for quantum computers to be worth deploying even if quantum advantage is never achieved. The relative usefulness of quantum computers depends on how the benefit of faster calculation compares to the higher cost of scaling. That does not imply that quantum computers will be immediately useful for a wide range of applications. Instead, the results suggest that quantum advantage is not the appropriate benchmark for a commercially viable quantum computer. Economics emphasizes decisions at the margins. Hence, fast calculations that are nevertheless feasible on classical computers may be the key to unlocking the potential of the quantum computing industry.

# References


Abhishek, Vibhanshu, Kinshuk Jerath, and Z. John Zhang (2016). "Agency Selling or Reselling? Channel Structures in Electronic Retailing". In: *Management Science* 62.8, pp. 2259–2280.

Adner, Ron, Jianqing Chen, and Feng Zhu (2020). "Frenemies in Platform Markets: Heterogeneous Profit Foci as Drivers of Compatibility Decisions". In: *Management Science* 66.6, pp. 2432–2451.

Anand, Krishnan S. and Manu Goyal (2009). "Strategic Information Management Under Leakage in a Supply Chain". In: *Management Science* 55.3, pp. 438–452.

Arute, Frank et al. (Oct. 2019). "Quantum supremacy using a programmable superconducting processor". In: *Nature* 7779, pp. 505–510. ISSN: 1476-4687.

Bakos, Yannis and Barrie Nault (1997). "Ownership and Investment in Electronic Networks". In: *Information Systems Research* 8.4.

Ball, Philip (2021). "First quantum computer to pack 100 qubits enters crowded race". In: *Nature* 599, p. 542.

Bova, Francesco, Avi Goldfarb, and Roger Melko (2021). "Commercial applications of quantum computing". In: *EPJ Quantum Technology* 8.1.

Bustamante, M. Cecilia and Laurent Frésard (2021). "Does Firm Investment Respond to Peers' Investment?" In: *Management Science* 67.8, pp. 4703–4724.

Choudhary, Vidyanand and Joseph Vithayathil (2013). "The Impact of Cloud Computing: Should the IT department be organized as a cost center or a profit center¿". In: *Journal of Management Information Systems* 30.2, pp. 67–100.

Cournot, Augustin and Irving Fisher (1929). *Researches into the mathematical principles of the theory of wealth; Augustin Cournot, 1838 ; translated by Nathaniel Bacon; with an essay on Cournot and mathematical economics.* [Macmillan Company].

Cusumano, Michael (2018). "Technology strategy and management: The business of quantum computing". In: *Communications of the ACM* 61.10, pp. 20–22.

Dellarocas, Chrysanthos, Zsolt Katona, and William Rand (2013). “Media, Aggregators, and the Link Economy: Strategic Hyperlink Formation in Content Networks”. In: *Management Science* 59.10, pp. 2360–2379.

Fazli, Amir, Amin Sayedi, and Jeffrey D. Shulman (2018). “The Effects of Autoscaling in Cloud Computing”. In: *Management Science* 64.11, pp. 5149–5163.

Grover, Lov K. (1996). “A Fast Quantum Mechanical Algorithm for Database Search”. In: *Proceedings of the Twenty-Eighth Annual ACM Symposium on Theory of Computing*. STOC ’96. Philadelphia, Pennsylvania, USA: Association for Computing Machinery, pp. 212–219.

Huang, Cupjin et al. (Sept. 2021). “Efficient parallelization of tensor network contraction for simulating quantum computation”. In: *Nature Computational Science* 1.9, pp. 578–587.

Kerenidis, Iordanis and Anupam Prakash (2017). “Quantum Recommendation Systems”. In: *8th Innovations in Theoretical Computer Science Conference (ITCS 2017)*. Ed. by Christos H. Papadimitriou. Vol. 67. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 49:1–49:21.

Moore, Gordon E. (2006). “Cramming more components onto integrated circuits, Reprinted from Electronics, volume 38, number 8, April 19, 1965, pp.114 ff.” In: *IEEE Solid-State Circuits Society Newsletter* 11.3, pp. 33–35.

Mueck, Leonie, Carmen Palacios-Berraquero, and Divya M Persaud (Feb. 2020). “Towards a quantum advantage”. In: *Physics World* 33.2, pp. 17–17.

National Academies of Sciences, Engineering and Medicine (2019). *Quantum Computing: Progress and Prospects*. Washington DC: The National Academies Press.

Niculescu, Marius F. and D.J. Wu (2014). “Economics of Free Under Perpetual Licensing: Implications for the Software Industry”. In: *Information Systems Research* 25.1, pp. 173–199.

Pan, Feng and Pan Zhang (2021). "Simulating the Sycamore quantum supremacy circuits". In: eprint: `arXiv:2103.03074`.

Ruane, Jonathan, Andrew McAfee, and William D. Oliver (2022). "Quantum Computing for Business Leaders". In: *Harvard Business Review* January-February.

Shor, Peter W. (1999). "Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer". In: *SIAM Review* 41.2, pp. 303–332.

Singh, Nirvikar and Xavier Vives (1984). "Price and Quantity Competition in a Differentiated Duopoly". In: *The RAND Journal of Economics* 15.4, pp. 546–554.

Tang, Ewin (2019). "A Quantum-Inspired Classical Algorithm for Recommendation Systems". In: *Proc. of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, pp. 217–228.

Track, Elie, Nancy Forbes, and George Strawn (2017). "The End of Moore's Law". In: *Computing in Science Engineering* 19.2, pp. 4–6.

Van Meter, Rodney and Clare Horsman (Oct. 2013). "A Blueprint for Building a Quantum Computer". In: *Commun. ACM* 56.10, pp. 84–93.

Wu, Yulin et al. (Oct. 2021). "Strong Quantum Computational Advantage Using a Superconducting Quantum Processor". In: *Phys. Rev. Lett.* 127 (18), p. 180501.

Yang, Jialei, Henry Chesbrough, and Pia Hurmelinna-Laukkanen (2021). "How to Appropriate Value from General-Purpose Technology by Applying Open Innovation". In: *California Management Review*, p. 00081256211041787.

Zhang, Xubing (2009). "Retailers' Multichannel and Price Advertising Strategies". In: *Marketing Science* 28.46, pp. 1080–1094.

Zhu, Feng and Marco Iansiti (2012). "Entry into Platform-Based Markets". In: *Strategic Management Journal* 33.1, pp. 88–106.

# 5 Proof of Lemma 1

For $\pi_1(q_1^*, q_2^*)$ and $\pi_2(q_1^*, q_2^*)$ to be concave in $x_1$ and $x_2$ respectively, and for $x_1^*$ and $x_2^* > 0$, $B_1$, $B_2$ and $a$ need to be sufficiently high. We plug the optimal quantity from each company from (3) and (4) into the profit functions in Equations (1) and (2) and take the second order derivatives with respective to $x_1$ and $x_2$ respectively. This yields the following outcomes:

$$\frac{\partial^2 \pi_1(q_1^*, q_2^*)}{\partial x_1^2} = \frac{8 - B_1(-4+\gamma^2)^2}{(-4+\gamma^2)^2} \tag{14}$$

$$\frac{\partial^2 \pi_2(q_1^*, q_2^*)}{\partial x_2^2} = \frac{8 - B_2(-4+\gamma^2)^2}{(-4+\gamma^2)^2} \tag{15}$$

For each respective second order condition to be negative, $B_1$ and $B_2$ need to be sufficiently high respectively. Specifically:

$$B_1 > \hat{\hat{B}}_1 = \frac{8}{(-4+\gamma^2)^2} \tag{16}$$

$$B_2 > \hat{\hat{B}}_2 = \frac{8}{(-4+\gamma^2)^2} \tag{17}$$

Thus, we assume (16) and (17) as they are necessary conditions for $\pi_1(q_1^*, q_2^*)$ ($\pi_2(q_1^*, q_2^*)$) to be concave in $x_1$ ($x_2$).

Next we assess the numerator for $x_1^*$ from (5). We define the numerator for $x_1^*$ as:

$$D = 4(a(4 - B_2(-2+\gamma)^2(2+\gamma)) + c(-4 + B_2(-2+c\gamma)(-4+\gamma^2))).$$

Taking the first derivative of $D$ with respect to $B_2$ we get

$$\frac{\partial D}{\partial B_2} = -4(-4+\gamma^2)(a(-2+\gamma) + c(2 - c\gamma)).$$

Note that $\partial D/\partial B_2 < 0$ for all $\gamma$ and for $a > c^2 > c > 1$. Solving for the value of $B_2$ which leads the numerator to equal zero will give us the value for $B_2$ over which the numerator is negative. For the numerator, that value is:

$$\hat{B}_2 = \frac{4(a-c)}{(-4+\gamma^2)(a(-2+\gamma)+c(2-c\gamma))}.$$

Thus, the numerator for $x_1^*$ will be negative if $B_2 > \hat{B}_2$.

Next we assess the numerator for $x_2^*$ from (6). We define the numerator for $x_2^*$ as:

$$E = 4(-4c^2 - B_1c(2c-\gamma)(-4+\gamma^2) + a(4 - B_1(-2+\gamma)^2(2+\gamma)))$$

Taking the first derivative of $E$ with respect to $B_1$ we get:

$$\frac{\partial E}{\partial B_1} = -4(c(2c-\gamma)+a(-2+\gamma))(-4+\gamma^2).$$

Note that $\partial E/\partial B_1 < 0$ for all $\gamma$ provided that $a > a^* = c(-2c+\gamma)/(-2+\gamma)$. Note also that this constraint approaches $a^* = c^2$ as $\gamma \to 0$. Thus, conditional on $a > a^*$, the numerator is decreasing in $B_1$. Solving for the value of $B_1$ which leads the numerator to equal zero will give us the value for $B_1$ over which the numerator is negative. For the numerator, that value is

$$\hat{B}_1 = \frac{4(a-c^2)}{(c(2c-\gamma)+a(-2+\gamma))(-4+\gamma^2)}$$

Thus, the numerator for $x_2^*$ will be negative if $B_1 > \hat{B}_1$ and $a > a^*$. Next, $x_1^*$ and $x_2^*$ both have the same denominator. We define this common denominator as:

$$F = -8(2 + B_2(-4+\gamma^2)) + B_1(-4+\gamma^2)(-8 + B_2(-4+\gamma^2)^2)$$

Taking the first derivative of $F$ with respect to $B_2$ we get:

$$\frac{\partial F}{\partial B_2} = -8(-4+\gamma^2) + B_1(-4+\gamma^2)^3$$

Note that $\partial F/\partial B_2 < 0$ provided $B_1 > \hat{\hat{B}}_1$. As we assume $B_1 > \hat{\hat{B}}_1$ in (16), $F$ is decreasing in $B_2$. Also

$$\frac{\partial F}{\partial B_1} = (-4+\gamma^2)(-8+B_2(-4+\gamma^2)^2).$$

Note that $\partial F/\partial B_1 < 0$ provided $B_2 > \hat{\hat{B}}_2$. As we assume $B_2 > \hat{\hat{B}}_2$ in (17), $F$ is also decreasing in $B_1$.

Next note that

$$\hat{B}_2 - \hat{\hat{B}}_2 = \frac{4(c(2c-\gamma)+a(-2+\gamma))\gamma}{(-4+\gamma^2)^2(a(-2+\gamma)+c(2-c\gamma))} > 0$$

if $a > a^*$, as previously assumed. Thus $\partial F/\partial B_1 < 0$ when $B_2 \geq \hat{B}_2$ provided $a > a^*$. Similarly,

$$\hat{B}_1 - \hat{\hat{B}}_1 = \frac{4\gamma(a(-2+\gamma)+c(2-c\gamma))}{(c(2c-\gamma)+a(-2+\gamma))(-4+\gamma^2)^2} > 0$$

if $a > a^*$. Thus $\partial F/\partial B_2 < 0$ when $B_1 \geq \hat{B}_1$ provided $a > a^*$. Finally, note that $F = 0$ when we set $B_1 = \hat{B}_1$ and $B_2 = \hat{B}_2$. As we assume that $a > a^*$, we know that $F$ is decreasing in $B_1$ when $B_1 \geq \hat{B}_1$ and $B_2$ when $B_2 \geq \hat{B}_2$. Thus $F < 0$ if $B_1 > \hat{B}_1$, $B_2 > \hat{B}_2$, and $a > a^*$.

Taken together, when $B_1 > \hat{B}_1$, $B_2 > \hat{B}_2$, and $a > a^* = c(-2c+\gamma)/(-2+\gamma)$, the numerators and denominators of both $x_1^*$ and $x_2^*$ are negative, and hence $x_1^*$ and $x_2^*$ are strictly positive.

In turn, $x_1^*, x_2^* > 0$ and $\pi_1(q_1^*, q_2^*)$ and $\pi_2(q_1^*, q_2^*)$ will be concave in $x_1$ and $x_2$, respectively, provided $B_1 > B_1^* = Max[\hat{B}_1, \hat{\hat{B}}_1]$, $B_2 > B_2^* = Max[\hat{B}2, \hat{\hat{B}}2]$, and $a > a^*$.

# 6 Proof of Proposition 1

Given the assumptions in Lemma 1, the quantum computing company is profitable for all $\gamma$. Optimal profits in (11) are:

$$\pi_1^* = \frac{B_1(-8+B_1(-4+\gamma^2)^2)(a(-4+B_2(-2+\gamma)^2(2+\gamma))+c(4-B_2(-2+c\gamma)(-4+\gamma^2)))^2}{(-8(2+B_2(-4+\gamma^2))+B_1(-4+\gamma^2)(-8+B_2(-4+\gamma^2)^2))^2}$$

Rearranging the terms in the equation above yields:

$$\pi_1^* = \frac{B_1(-8 + B_1(-4+\gamma^2)^2)x_1^{*2}}{16}$$

Given the assumptions in Lemma 1, $x_1^{*2} > 0$. Additionally, as we assume in Equation (16) in Lemma 1, we know that $B_1 > \hat{\hat{B}}1 = 8/(-4+\gamma^2)^2 > 0$ and thus the first two terms in the numerator are also positive. Thus, optimal profits are positive for all $\gamma$ given the assumptions made in Lemma 1.

# 7 Proof of Proposition 2

We generate the conditions where $x_1^* > x_2^*$. First, we calculate $x_1^* - x_2^*$. Doing so yields the following expression:

$$\frac{4(4(-1+c)c - B_2(-4+\gamma^2)(a(-2+\gamma)+c(2-c\gamma)) + B_1(c(2c-\gamma)+a(-2+\gamma))(-4+\gamma^2))}{-8(2+B_2(-4+\gamma^2)) + B_1(-4+\gamma^2)(-8+B_2(-4+\gamma^2)^2)}.$$

We note that given the assumptions in Lemma 1, the denominator of the expression above is negative. Thus for $x_1^* > x_2^*$ the numerator also needs to be negative. Notably:

- $4(-1+c)c > 0$ as we assume $c > 1$
- $-B_2(-4+\gamma^2)(a(-2+\gamma)+c(2-c\gamma)) < 0$ as we assume $a > c^2 > c > 1$
- $+B_1(c(2c-\gamma)+a(-2+\gamma))(-4+\gamma^2)) > 0$ as we assume $a > a^* = c(-2c+\gamma)/(-2+\gamma)$ is a necessary condition to get $x1^*, x1^* > 0$.

Thus $x_1^* > x_2^*$ if

$$4(-1+c)c + B_1(c(2c-\gamma)+a(-2+\gamma))(-4+\gamma^2) < B_2(-4+\gamma^2)(a(-2+\gamma)+c(2-c\gamma))$$

or,

$$\frac{4(-1+c)c+B_1(c(2c-\gamma)+a(-2+\gamma))(-4+\gamma^2)}{(-4+\gamma^2)(a(-2+\gamma)+c(2-c\gamma))}<B_2$$

Finally, we illustrate via a numerical example that there are instances where $x_1^* > x_2^*$ even when $B_1 > B_2$ that do not violate any of the assumptions in the paper including those in Lemma 1: We set $B_1 = 10$, $B_2 = 8$, $c = 2$, $\gamma = 1$, $a = 20$ (similar to the parameters in our figures). With these inputs:

$$B_1 = 10 > B_2 = 8,$$

$$\frac{4(-1+c)c+B_1(c(2c-\gamma)+a(-2+\gamma))(-4+\gamma^2))}{(-4+\gamma^2)(a(-2+\gamma)+c(2-c\gamma))}=\frac{107}{15}<B_2=8,$$

$$x_1^*=\frac{102}{109}>x_2^*=\frac{89}{109}$$